False Alarm Control for Window-Limited Change Detection

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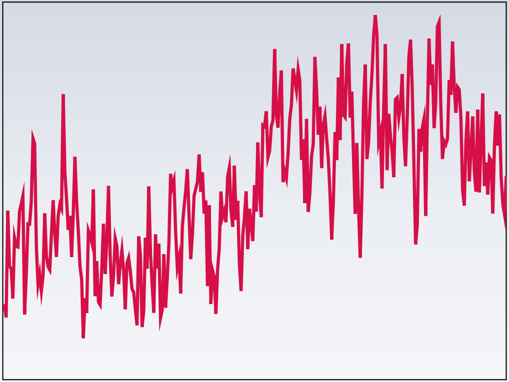
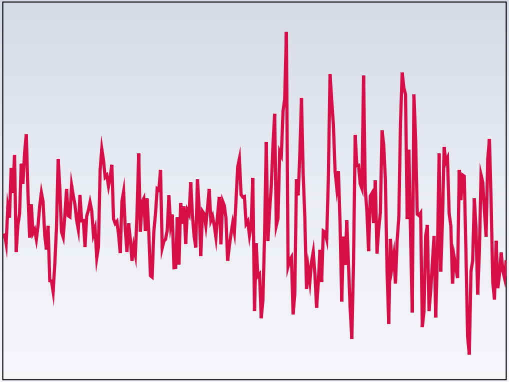
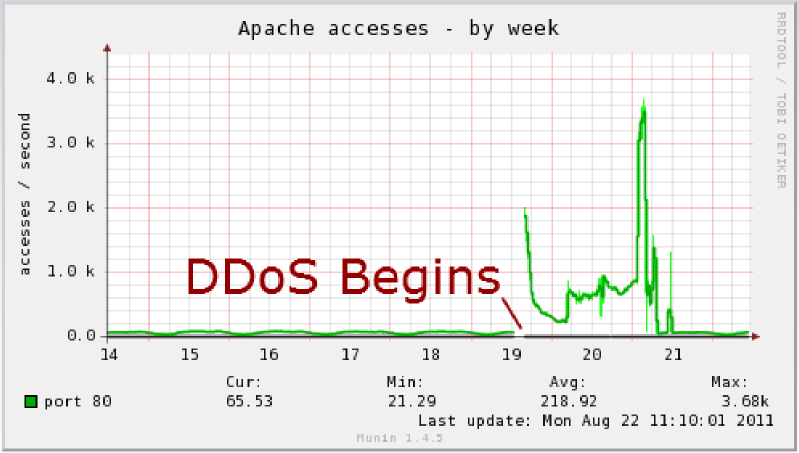
Change in mean



Change in variance



Why is that interesting?



Outline

Change Detection Procedures

Unconditional Local False Alarm Probability

Maximum Local False Alarm Probability

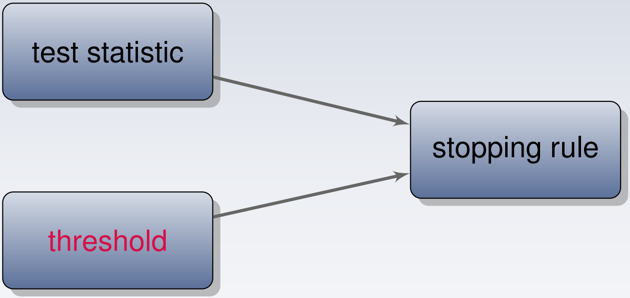
Window Limited False Alarm Probability

We need to decide whether a change has occurred or not:

H_0 : No change has occurred.

H_1 : There is a change point k with $1 \leq k \leq m$.

How do we do that?



Traditional approach

Test statistic: $\mathrm{LLR}_{k,m} = \sum_{t=k}^{m} \ell(X_t)$ (CUSUM)

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Threshold:	Asymptotics (for i.i.d.), recursive integral equations, numerical methods

Optimality

Minimises detection delay among all procedures that satisfy

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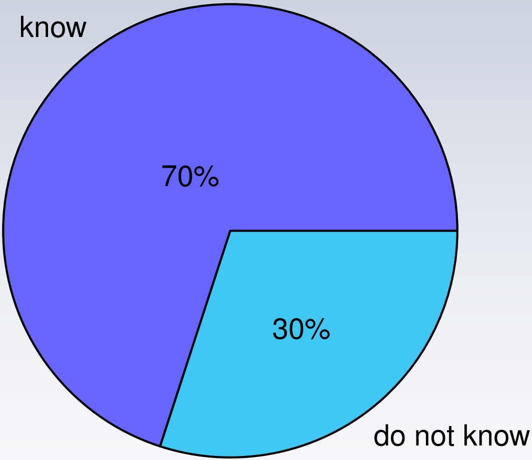
- $\mathbb{E}_0\tau \geq \kappa$ as $\kappa \to \infty$ (asymptotic optimality)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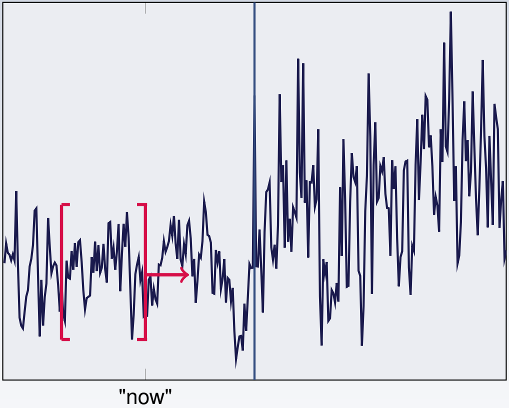
Procedure



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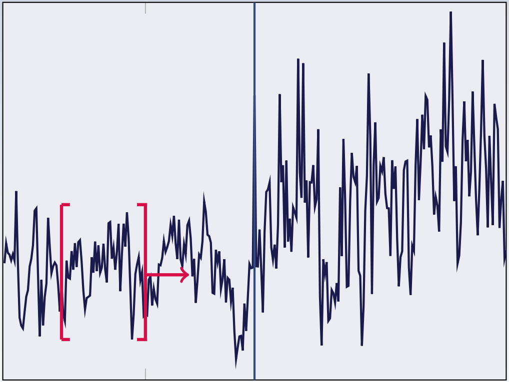

$$\tau = \inf\left\{m : \max_{\{1\leq k\leq m\}} \mathrm{LLR}_{k,m} > b\right\}$$

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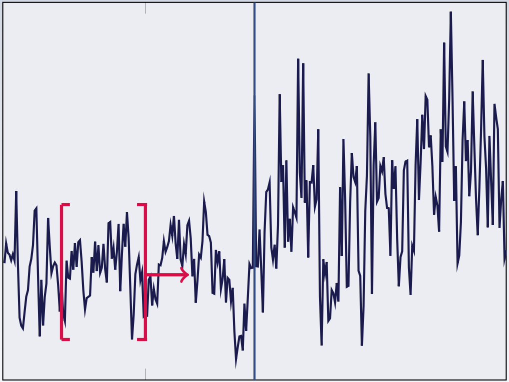

$$\tau_{WL} = \inf \left\{ m : \max_{\{m-n \leq k \leq m\}} \mathrm{LLR}_{k,m} > b \right\}$$

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Asymptotics

$$\mathrm{ULFA}_n := \sup_{m \geq 0} \mathbb{P}_0(m \leq \tau \leq m + n)$$

- CUSUM is asymptotically optimal for small α among procedures that satisfy $\mathrm{ULFA}_n \leq \alpha$ (Lai, 1998).

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- **But:** For large thresholds (small α) this is decreasing in m (Tartakovsky, 2005).

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Maximum local FA

$$\begin{aligned}\mathrm{MLFA}_n &:= \sup_{m\geq 0} \mathbb{P}_0\left(\tau \leq m+n \,|\, \tau > m\right)\\ &= \sup_{m\geq 0} \frac{\mathbb{P}_0(m < \tau \leq m+n)}{\mathbb{P}_0(\tau > m)}\end{aligned}$$

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Local FA probability:	$\sup_{m \geq 0} \mathbb{P}_0(\tau \leq m + n \mid \tau \geq m)$
Unconditional local FA prob.:	$\sup_{m \geq 0} \mathbb{P}_0(m \leq \tau \leq m + n)$

Asymptotic optimality?

If $\mathbb{P}_0\big(\tau_{WL} \leq N\big) \leq \alpha$, then:

(i) The ARL condition $\mathbb{E}_0\tau \geq \kappa$ is satisfied for $\kappa = (n+1)(1-\alpha)$.

(ii) The ARL condition $\mathbb{E}_0\tau_{WL} \geq \kappa_{WL}$ is satisfied for $\kappa_{WL} = \max\{N(1-\alpha), \kappa\}$.

How we may want to think about it

Define

$$S_{k:n}(m) := \sum_{i=k+m}^{n+m} \ell(X_i)\,.$$

Then

$$\mathbb{P}_0\left(\tau_{WL} \le N\right) = \mathbb{P}_0\big(\boldsymbol{M}_N > b\big)\,,$$

where

$$\boldsymbol{M}_N := \begin{pmatrix} \max\left\{S_{1:n}(0), \ldots, S_{1:n}(N)\right\} \\ \vdots \\ \max\left\{S_{n:n}(0), \ldots, S_{n:n}(N)\right\} \end{pmatrix}.$$

Get a grip on $\boldsymbol{M}_n$

For many popular CP test statistics we can write

$$\boldsymbol{Y}_m = C\boldsymbol{Y}_{m-1} + \vartheta \mathbf{1} \ell(X_{m+n}),$$

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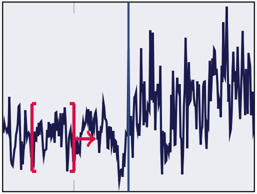
Example: CUSUM

$$S_{k:n}(m) = S_{k:n}(m-1) - \ell(X_{k+m-1}) + \ell(X_{n+m})$$



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Example: CUSUM

$$\boldsymbol{Y}_m = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{Y}_{m-1} + \mathbf{1}\ell(X_{m+n})$$

Other examples:

Moving average, EWMA, non-parametric CUSUM

Ugly integral equation

Let, for fixed $\boldsymbol{x} \in \mathbb{R}^n$,

$$Q_m(\boldsymbol{x}) := \mathbb{P}\left(\boldsymbol{M}_m \le \boldsymbol{x}\right)$$

for $m \ge 0$.

Proposition

We have $Q_m(\boldsymbol{x}) = \mathcal{K}\, Q_{m-1}(\boldsymbol{x})$ for $m \ge 0$, where

$$\mathcal{K}\, h(\boldsymbol{x}) = \int_{\mathbb{R}} F\left(\frac{1}{\vartheta} \min_{i=1,\dots,n} \left\{\left(\boldsymbol{x} - \Psi(\boldsymbol{z})\right)_i\right\}\right) \mathrm{d}h(\boldsymbol{z})\,.$$

Numerical solution: Withers and Nadarajah (2014)

Extreme value distribution

Theorem (Amram, 1985)

For standard stationary Gaussian random vectors $\boldsymbol{Y}_m$ that are "not too dependent", and large m:

$$\mathbb{P}(\boldsymbol{M}_{m,i} \leq a_m \boldsymbol{x}_i + c_m,\ i = 1, \ldots, n) \approx \prod_{i=1}^{n} \exp\left(-\exp(-\boldsymbol{x}_i)\right),$$

(we know what a_m and c_m are).

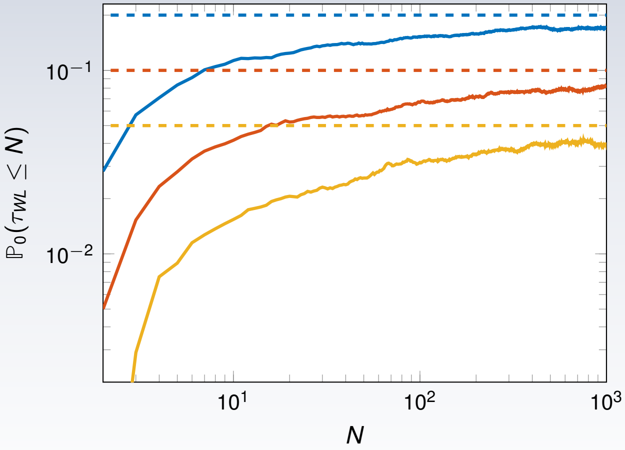
Set the threshold to be the constant

$$b_N := -a_N \log\left(-\frac{1}{n}\log(1-\alpha)\right) + c_N\,,$$

for fixed N and n.

Example

Dashed lines: $\alpha \in \{0.05, 0.1, 0.2\}$



Gaussian approximation

$$\mathbb{P}_0\left(\max_{1\le k\le n} \mathrm{LLR}_{k,n} > b\right) \approx \mathbb{P}_0\left(\max_{t\in[0,n]} \sigma B_t + \mu\, t \ge b\right)$$
$$= 1 - \Phi\left(\frac{b-\mu\, n}{\sigma\sqrt{n}}\right) + e^{\frac{2b\mu}{\sigma^2}}\,\Phi\left(\frac{-b-\mu\, n}{\sigma\sqrt{n}}\right)$$

Example: State space model

$$X_{t+1} = AX_t + Y_t + M\,\mathbb{1}_{\{t \geq k\}}$$

$$V_t = BX_t + Z_t + N\,\mathbb{1}_{\{t \geq k\}}$$

Large deviation approximation

The Gärtner Ellis theorem suggests to pick the threshold **function** b such that it satisfies

$$\alpha = \exp\Big(-n\mathcal{I}\big(b(\beta)\big)\Big)$$

for all $0 \leq \beta \leq 1$ such that $n\beta + 1$ integer, where we can compute

$$\mathcal{I}\big(b(\beta)\big) = \sup_{\lambda} \left\{ \lambda b(\beta) + (1-\beta)\frac{\lambda}{2}(1-\lambda)\rho'\Omega^{-1}\rho \right\} .$$

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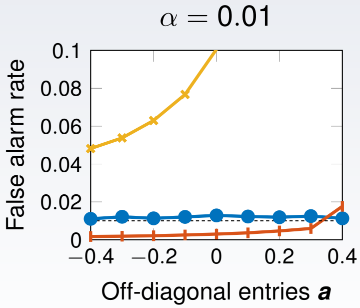
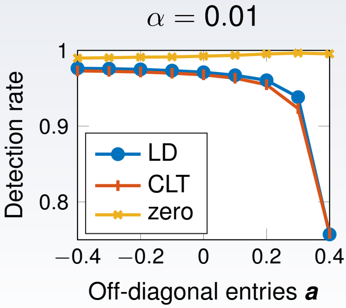
$=>$ Simple closed form expression for $b(\beta)$.

Example: State space model

$$X_{t+1} = \begin{pmatrix} 0.5 & \boldsymbol{a} \\ \boldsymbol{a} & 0.5 \end{pmatrix} X_t + Y_t + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbb{1}_{\{t \geq k\}}$$

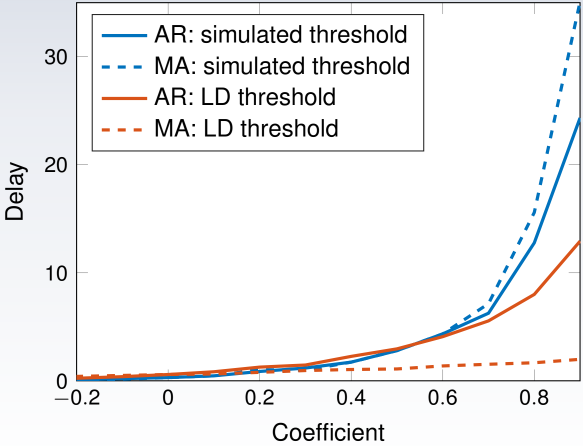
$$V_t = BX_t + Z_t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \mathbb{1}_{\{t \geq k\}}$$

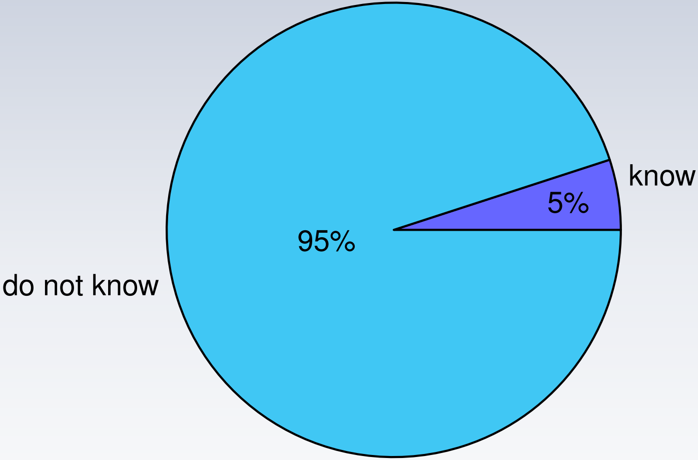




Example: ARMA model

$$\text{AR:} \quad X_t = \varphi X_{t-1} + \varepsilon_t$$
$$\text{MA:} \quad X_t = \vartheta \varepsilon_{t-1} + \varepsilon_t$$

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Some References

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